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THE THEORY OF IDEAL YANG-MILLS FLUIDS IN SYMMETRIC HYPERBOLIC FORM *

Maurice H.P.M. van Putten
Institute for Theoretical Physics
UCSB, Santa Barbara, CA 93106, USA

Abstract

The theory describing the evolution of perfect fluids coupled to a Yang-Mills field in the presence of infinite electrical conductivity (ideal Yang-Mills fluids) is proved to be hyperbolic. This result is obtained by embedding the theory in a symmetric hyperbolic system of equations.

A divergence formulation of the theory of ideal Yang-Mills fluids forms the basis of our analysis. This divergence form constitutes a covariant and constraint-free formulation. The infinitesimally small amplitude wave structure is derived, and found to be determined by a sixth-order polynomial. Alfven waves ($\delta P = \delta r = 0$) do not, in general, exist, and occur in very special cases only. The theory is embedded in a symmetric hyperbolic system using a generalization of Friedrichs symmetrization procedure. This establishes at once both algebraic hyperbolicity (all wave speeds real and less than unit) and dependence on initial data to within finite order.

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1 Introduction

A theory of Yang-Mills fluids has recently been discussed in the limit of infinite conductivity, *ideal Yang-Mills fluids* (IYMF), by Choquet-Bruhat [**?**, **?**]. This describes relativistic fluids with a colored magnetic fields. In this paper, we concentrate on the mathematical structure of such Yang-Mills fluids. The main purpose is to establish well-posedness of initial value problems in this theory. Specifically, it will be established

Theorem 1 *Consider relativistic hydrodynamics in which initial value problems are well posed by suitable equation of state. For a fluid with infinite conductivity, well-posedness is preserved in the presence of a Yang-Mills field.*

It is unknown whether a field description for colored magnetic fields is physically appropriate to describe the dynamics of high density matter at finite temperature, in view of our uncertainty about screening of colored magnetic fields. The well-posedness result Theorem 1 above shows that weak screening of colored magnetic fields, though uncertain, is permissable.

We also derive expressions for the infinitesimally small amplitude wave structure. While finite conductivity decouples the hydrodynamical and "electromagnetic waves," this is no longer true in the singular limit of infinite conductivity.

The analysis is based on two reformulations of the theory of IYMF. We begin by obtaining the theory of ideal Yang-Mills fluids in *divergence form* following [**?**, **?**]. This formulation is fully covariant and constraint-free. It contains the Yang-Mills solutions by dynamically conserving the

original flux-freezing constraints. This formulation Cauchy-Kowalewski regularizes the original formulation of IYMF (*cf.* [?]) and, therefore, defines the characteristic determinant. Continuing, we establish well posedness through symmetrization. Friedrichs & Lax [?] and Friedrichs [?] introduced a symmetrization procedure for systems of conservation laws in the presence of a "main dependency relation" and a certain convexity property. In this paper, a generalization of their symmetrization procedure is presented which is appropriate for the divergence formulation of ideal Yang-Mills fluids. This symmetrization procedure allows for the theory of ideal Yang-Mills fluids to be embedded in symmetric hyperbolic form. Given this, the standard regularity results for quasi-linear symmetric hyperbolic systems apply (see, in particular, Fisher & Marsden [?] and referenced therein), which establishes dependence on initial data to within finite order.

The organisation of this paper is as follows. In Section 2, the theory of ideal Yang-Mills fluids with infinite conductivity is stated. This theory is formulated in covariant, constraint-free divergence form in Section 3. The characteristic determinant is derived in Section 4. A symmetrization procedure in the presence of conserved constraints is presented in Section 5. Using this symmetrization procedure, ideal Yang-Mills in divergence form is shown to allow for an embedding in a symmetric hyperbolic system (Section 6).

2 Ideal Yang-Mills fluids

The theory of Yang-Mills fluids describes the evolution of a compressible fluid possessing a magnetic field, h_a, with values in a non-abelian Lie algebra. In

the presence of infinite conductivity, the magnetic field is governed by the Yang-Mills' generalization of Faraday's induction law. This description may be given in the presence of a curved space-time background with prescribed hyperbolic metric g_{ab} (thereby considering Yang-Mills *test* fluids). We will do so with metric signature $(-,+,+,+)$.

The fluid to be considered is assumed to be inviscid and compressible with given equation of state. For example, the fluid may be modelled with a polytropic equation of state, relating the isotropic hydrostatic pressure, P, to the rest mass density, r, as

$$P = Kr^{\gamma}. \tag{1}$$

Here, K is the adiabatic constant and γ is the polytropic index. K is conserved along the fluid's world lines, except across shocks. For a discussion of thermodynamics of relativistic fluids in some generality, see Lichnerowicz [**?**]. Let the velocity four-vector field u^a represent the tangent to the world lines of the fluid elements. Then $U = (u^a, r, P)$ decribes the state of the fluid elements, and is governed by conservation of energy-momentum and baryon number:

$$\begin{cases} \nabla_a T_f^{ab}(U) = 0, \\ \nabla_a (r u^a) = 0, \\ c_0(U) := u^2 + 1 = 0. \end{cases} \tag{2}$$

Here, $T_f^{ab} = rfu^a u^b + Pg^{ab}$ is the stress-energy tensor for a perfect fluid with the specific enthalpy, f. For example, $f := 1 + \frac{\gamma}{\gamma-1}\frac{P}{r}$ in case of a polytropic equation of state (1). Notice that (2) constitutes six equations in the six unknowns U.

The theory of Yang-Mills fluids in the presence of infinite conductivity describes the dynamic interation between a perfect fluid and a (charged) magnetic field given by a one-form, h_a, with its values in a non-abelian Lie algebra $\mathcal{G}$ of dimension N. $\mathcal{G}$ possesses totally antisymmetric structure constants $c_{\lambda\mu\nu}$ (see, *e.g.*, [**?**]). Elements from $\mathcal{G}$ will be indexed in Greek. We thus consider a problem in $6+4N$ variables $U = (u^a, h_a, r, P)$. To be precise, the Yang-Mills $\mathcal{G}$-valued 2-form

$$F_{ab} := \nabla_a A_b - \nabla_b A_a + [A_a, A_b] \tag{3}$$

in the presence of infinite conductivity,

$$u^a F_{ab} \equiv 0, \tag{4}$$

may be expressed in terms of a $\mathcal{G}$-valued 1-form h_a as

$$F_{ab} = \epsilon_{abcd} h^c u^d. \tag{5}$$

Here, ϵ_{abcd} is the natural volume element on the space-time manifold. The condition $u^c h_c = 0$ further imposes uniqueness on this representation (5) of the Yang-Mills field. The evolution of h_a is governed by the so-called *Yang-Mills equations* (see [**?**, **?**] for a derivation)

$$K : \begin{cases} \hat{\nabla}_a \{h^{[a} u^{b]}\} = 0, \\ c(U) := h^c u_c = 0. \end{cases} \tag{6}$$

Here, the gauge invariant derivative $\hat{\nabla}_a := \nabla + [A, \cdot]$ is introduced so as to ensure gauge invariance in (6) (*cf.* [**?**, **?**]). The conditions $c_\lambda(U) := h^c_\lambda u_c = 0$ constitute N constraints, one on each Lie-element h^a_λ, and may be considered

the *flux-freezing* constraints in Yang-Mills fluids following the nomenclature in ideal magneto-hydrodynamics. They impose the condition that the magnetic field be purely spatial in the fluid's rest frame.

The magnetic field-fluid coupling is established by nonzero magnetic field energy and stresses. In the limit of infinite conductivity, this amounts to adding the stress-energy tensor T_m^{ab},

$$T_m^{ab}(U) = h^2 u^a u^b + \frac{1}{2} h^2 g^{ab} - h^a h^b, \tag{7}$$

to the fluid stress-energy tensor, $T_f^{ab}(U)$, mentioned above. Here, summation over the Lie-index is implicit. In anticipation of what follows, it becomes appropriate to define the *Yang-Mills magnetic stress tensor* $H_b^a := h^a h_b \equiv \sum_\lambda h_\lambda^b h_{\lambda a}$. Notice that $T_m^{ab}(U)$ is entirely determined by H_a^b, as $h^2 = H_c^c$. The evolution of Yang-Mills fluids in the presence of infinite conductivity as a test fluid is now described by [?, ?]

$$\begin{cases} \nabla_a T^{ab} = 0, \\ \hat{\nabla}_a \{h^{[a} u^{b]}\} = 0, \\ \nabla_a (r u^a) = 0, \\ c_\lambda(U) = 0, \\ c_0(U) = 0, \end{cases} \tag{8}$$

where $T^{ab}(U) = T_f^{ab}(U) + T_m^{ab}(U)$ is the total stress-energy tensor. Notice that (8) constitutes essentially $6 + 4N$ equations. Of course, (8) must be closed by (3) and (5).

It is imperative to observe that (8) constitutes a partial differential-algebraic system of equations. Without the constraints $c_\lambda(U) = c_0(U) = 0$ the remaining partial differential equations in (8) do not impose uniqueness on the evolution of the unknowns U. In other words, (8) is not regular in the

sense of Cauchy-Kowalewski (*cf.* [**?**]). On the other hand, systems of purely partial-differential equations are the most commonly studied, both analytically and numerically. It now becomes of interest to first obtain the theory of ideal Yang-Mills fluids in covariant form as a system of precisely this type with no constraints.

In what follows we will refer to the theory described by (8) as the theory of *ideal* Yang-Mills fluids, analoguous to the nomenclature *ideal* magnetohydrodynamics in the case of a single magnetic field ($N = 1$) with infinite 'electrical' conductivity.

3 Ideal Yang-Mills in divergence form

Consider the initial value problem for ideal Yang-Mills fluids. Maxwell's equations require initial data, $U^{(0)}$, to satisfy certain compatibility conditions. Let Σ denote an initial, space-like submanifold with (time-like) normal one-form, ν_a, and write

$$\hat{\nabla}_a = -\nu_a(\nu_c\hat{\nabla}^c) + (\hat{\nabla}_\Sigma)_a \tag{9}$$

for $\hat{\nabla}_a$ on Σ. The initial data $U^{(0)}$ are then subjected to the conditions (*cf.* [**?**] for $N = 1$)

$$\left\{ \begin{array}{l} \nu_b(\hat{\nabla}_\Sigma)_a\{h^{[a}u^{b]}\} = 0, \\ c_\lambda(U) = 0. \end{array} \right. \tag{10}$$

This constitutes a covariant formulation of the condition that the initial magnetic field be divergence free in the space-like, ν_a-orthogonal submanifold Σ, and that the magnetic field is purely space-like for observers whose world-lines have tangents u^a.

Application of the technique described in [?] to each K_λ, $\lambda = 1, \cdots, N$, yields [?]

Theorem 3.1 *The equations of ideal Yang-Mills fluids including the flux-freezing constraints can be stated as*

$$\hat{\nabla}_a F^{aA} \equiv \begin{cases} \nabla_a T^{ab} = 0, \\ \hat{\nabla}_a \{ h^{[a} u^{b]} + g^{ab} c(U) \} = 0, \\ \nabla_a (r u^a) = 0, \\ \nabla_a \{ (u^c u_c + 1) \xi^a \} = 0. \end{cases} \tag{11}$$

This system must be closed by

$$\nabla_a A_b - \nabla_b A_a = \epsilon_{abcd} h^c u^d - [A_a, A_b]. \tag{12}$$

The vector ξ^a is any prescribed time-like vector field. This system is equivalent to (8) in regions where the flow is continuously differentiable. The standard jump conditions across surfaces of discontinuity for this system are those of conservation of energy-momentum, baryon number and Yang-Mills equations.

Proof : Clearly, we need only show that a solution to an initial value problem in the new formulation with Cauchy-data satisfying compatibity conditions (10) on an initial, space-like hypersurface Σ, yields a solution to the original system of PDAE's (8). We will do so by showing that c_λ satisfies the a homogeneous wave equation with vanishing Cauchy-data:

$$\begin{cases} \Box c = 0 & \text{in } D^+(\Sigma), \\ c = 0 & \text{on } \Sigma, \\ \nu^a \nabla_a c = 0 & \text{on } \Sigma. \end{cases} \tag{13}$$

Here, we have defined the *Yang-Mills wave operator* $\Box = \nabla^c \nabla_c + [(\nabla^c A_c), \cdot] + [A^c \nabla_a, \cdot] + [A^c, [A_c, \cdot]]$. $D^+(\Sigma)$ denotes the future domain of dependence

of Σ, and ν_a denotes the normal one-form to Σ. In what follows, $\epsilon_{abcd} = \sqrt{-g}[abcd]$ will denote the volume element on our four-dimensional manifold with $\sqrt{-g} = \det[g_{ab}]$. Here, $[abcd]$ is the totally antisymmetric symbol such that $[0123] = 1$.

Ricci's identity implies the identity [**?**]

$$\hat{\nabla}_a \hat{\nabla}_b (2h^{[a}u^{b]}) + \frac{1}{4}[F_{ab}, F_{cd} - \epsilon_{cdef}h^e u^f]\epsilon^{abcd} \equiv 0. \tag{14}$$

By (12), $F_{ab} = \epsilon_{abcd}h^c u^d$ is enforced, so that $\hat{\nabla}_a \hat{\nabla}_b (2h^{[a}u^{b]}) \equiv 0$. Consequently, we are left with the Yang-Mills wave equation for c:

$$0 = \hat{\nabla}_a \hat{\nabla}_b (\omega^{ab}) = \hat{\nabla}_a \hat{\nabla}_b (h^{[a}u^{b]}) + (\hat{\nabla}^c \hat{\nabla}_c)c = \Box c. \tag{15}$$

On Σ, we may write

$$\hat{\nabla}_a = -\nu_a(\nu^c \hat{\nabla}_c) + (\hat{\nabla}_\Sigma)_a. \tag{16}$$

Using this, Cauchy-data satisfying (10) yield

$$\begin{aligned} 0 &= \nu^b\{\hat{\nabla}^a(h_{[a}u_{b]} + g_{ab}c)\} \\ &= -\nu^b\nu^a(\nu^c\hat{\nabla}_c)\{h_{[a}u_{b]}\} + \nabla^b(\hat{\nabla}_\Sigma)^a\{h_{[a}u_{b]}\} + \nu^b\hat{\nabla}_b c \\ &= \nu^b\nabla_b c, \end{aligned} \tag{17}$$

because $h_{[a}u_{b]}$ is antisymmetric.

This forces $c \equiv 0$ in $D^+(\Sigma)$, and the proof is complete. $\Box$

We remark that for isentropic fluids, where the entropy is constant everywhere, the fluid variables r and P depend on essentially one parameter, in view of $\mathrm{d}P|_S = r\mathrm{d}f$. In this event, conservation of energy-momentum, $\nabla_a T^{ab}(U) = 0$, together with conservation of baryon number, $\nabla_a(ru^a) = 0$,

may be seen to yield conservation of the constraint $c_0(U) = 0$. (11) then reduces to

$$\left\{ \begin{array}{c} \nabla_a T^{ab} = 0, \\ \hat{\nabla}_a \{ h^{[a} u^{b]} + g^{ab} c(U) \} = 0, \\ \nabla_a (r u^a) = 0. \end{array} \right. \tag{18}$$

(11) strictly contains ideal Yang-Mills solutions to PDAE (8). Ideal Yang-Mills solutions are characterized by vanishing of the flux-freezing constraints

$$c_\lambda(U) = 0. \tag{19}$$

Initial data satisfying the compatibility conditions (10) ensure that the $c_\lambda(U)$ are preserved as zero (as in the case of MHD [**?**]). The formulation of the constraint $c_0(U)$ in terms of the last equation in (11) is taken from the divergence formulation of ideal MHD [**?**].

4 Ideal Yang-Mills' characteristic determinant

Infinitesimally small amplitude waves may be defined through the normal cone, $\mathcal{N}$, of the one-forms ν_a normal to their time-like surfaces of propagation. We will consider waves whose velocity of propagation is strictly less than unity, *i.e.*, waves with $\nu^c \nu_c \neq 0$. The condition on $\nu_a \epsilon \mathcal{N}$ is given by the vanishing of the characteristic determinant (pointwise on the space-time manifold)

$$D(U; \nu_a) := \det \frac{\partial F^{aA} \nu_a}{\partial U^B} = 0. \tag{20}$$

Here, B indexes the unknowns $U = (u^a, h_a, P, r)$, as defined before. Clearly, $D(U; \nu_a)$ constitutes a $6 + 4N$ degree polynomial in ν_a. The wave struc-

ture may be classified by finding the factorization of $D(U;\nu_a)$ over miminal polynomials in ν_a.

Using Theorem 3.1, we have

$$D(U;\nu_a) = \det \begin{pmatrix} K & R_1 & \cdots & & R_N & |* \\ \hline \\ S^1 & A & & & & |0 \\ S^2 & & A & & & |0 \\ \cdots & & & \cdots & & \\ S^N & & & & A & |0 \\ \hline \\ * & 0 & \cdots & & 0 & |* \end{pmatrix}. \tag{21}$$

Here, we have introduced the 4 by 4 matrices (when written in component form)

$$\begin{aligned} &K_a^b := \tfrac{\partial T^{cb}\nu_c}{\partial u^a} = (rf+h^2)(u^c\nu_c)\delta_a^b + (rf+h^2)u^b\nu_a, \\ &R_{\lambda a}^b := \tfrac{\partial T^{cb}\nu_c}{\partial h_\lambda^a} = 2(u^c\nu_c)u^b h_{\lambda a} + \nu^b h_{\lambda a} - h_\lambda^b\nu_a - (h_\lambda^c\nu_c)\delta_a^b, \\ &S_a^{\lambda b} := \tfrac{\partial \omega_\lambda^{cb}\nu_a}{\partial u^a} = -h_\lambda^b\nu_a + (h_\lambda^c\nu_c)\delta_a^b + \nu^b h_{\lambda a}, \\ &A_a^b := \tfrac{\partial \omega_\lambda^{cb}\nu_c}{\partial h_\lambda^a} = -(u^c\nu_c)\delta_a^b + u^b\nu_a + \nu^b u_a, \end{aligned} \tag{22}$$

where $\omega_\lambda^{ab} := h^a u_\lambda^b - h^b u_\lambda^a + g^{ab}c_\lambda(U)$. The main step towards factorization is contained in

Proposition 4.1 *The Yang-Mills characteristic determinant $D(U;\nu_a)$ contains the factor $|A|^{N-1}$.*

The proof of the Proposition is based on a suitable rewriting of the characteristic matrix $[F_A^B](U;\nu_a)$ in (20) and (21), and the observation

Lemma 4.1 *Consider a rank-one update, M_λ, to an $N\times N$ matrix, $M = [m_a^b]$, through an N-dimensional column vector, u^b, and row vector, v_a, of the form $M+\lambda[u^b v_a]$. The determinant of $M_\lambda = M+\lambda[u^b v_a]$ is linear in λ.*

Proof : If either u^b or v_a is zero, there is nothing to prove. Let, therefore, $v_k \neq 0$. We have

$$\begin{aligned}
\det M_\lambda &= \det\, [[m_1^b + \lambda u^b v_1], \cdots, [m_N^b + \lambda u^b v_N]] \\
&= \det\, [[m_1^b - \tfrac{v_1}{v_k} m_k^b], \cdots, [m_{k-1}^b + \lambda u^b v_{k-1}], [m_k^b + \lambda u^b v_k], \\
&\quad [m_{k+1}^b + \lambda u^b v_{k+1}], \cdots, [m_N^b - \tfrac{v_N}{v_k} m_N^b]] \\
&\equiv \det\, [[\bar{m}_1^b], \cdots, [\bar{m}_{k-1}^b], [m_k^b + \lambda u^b v_k], [\bar{m}_{k+1}^b], \cdots, [\bar{m}_N^b]].
\end{aligned} \tag{23}$$

Because of the definition $\det\, [H_a^b] = \sum_\sigma (-1)^{\epsilon_{\sigma_1 \cdots \sigma_N}} H^1_{\sigma_1} H^2_{\sigma_2} \cdots H^N_{\sigma_N}$ for the determinant of an $N \times N$ matrix $[H_a^b]$, where ϵ is the $N-$dimensional totally antisymmetric symbol and the sum is over all N-permutations σ, it follows that each term in our determinant contains λ^k with $k = 0, 1$. $\Box$

Proof of Proposition 4.1: The result will follow in several steps. The first two steps concern rewriting of the characteristic matrix of Yang-Mills' in divergence form through column block operations by (unitary matrix) multiplications from the *right* and row block operations by (unitary matrix) multiplications from the *left*. The final result follows from further inspection of a resulting 6 by 6 reduced matrix.

Step (a): Operate on the leading column block through multiplication by matrices with unit determinant from the *right*:

$$D(U;\nu_a) = [\frac{\partial F^{aA}\nu_a}{\partial U^B}] \left(\begin{array}{ccccc|c}
\mathrm{id} & 0 & 0 & \cdots & 0 & 0 \\ \hline
-A^{-1}S^1 & \mathrm{id} & & & & 0 \\
0 & & \mathrm{id} & & & 0 \\
\cdots & & & \cdots & & \\
0 & & & & \mathrm{id} & 0 \\ \hline
0 & 0 & \cdots & & 0 & \mathrm{id}
\end{array} \right) \tag{24}$$

$$
= \det \left(\begin{array}{ccccc|c}
K - R_1 A^{-1} S^1 & R_1 & \cdots & & R_N & * \\
\hline
0 & A & & & & 0 \\
S^2 & & A & & & 0 \\
\cdots & & & \cdots & & \\
S^N & & & & A & 0 \\
\hline
* & 0 & \cdots & & 0 & *
\end{array} \right). \tag{25}
$$

Proceeding in this manner to cancel each S^λ in the leading column block through subsequent multiplication of the matrix in (25) from the right by matrices with unit determinant of the form

$$
\left(\begin{array}{ccccccc|c}
\mathrm{id} & 0 & \cdots & & & & & 0 \\
\hline
0 & \mathrm{id} & & & & & & 0 \\
\cdots & & & \cdots & & & & \\
0 & & \mathrm{id} & & & & & 0 \\
-A^{-1} S^\lambda & & & \mathrm{id} & & & & 0 \\
0 & & & & \mathrm{id} & & & 0 \\
\cdots & & & \cdots & & & & \\
0 & & & & & \mathrm{id} & & 0 \\
\hline
0 & \cdots & & & & & & \mathrm{id}
\end{array} \right), \tag{26}
$$

we obtain

$$
D = \det \left(\begin{array}{ccccc|c}
K - R A^{-1} S & R_1 & \cdots & & R_N & * \\
\hline
0 & A & & & & 0 \\
0 & & A & & & 0 \\
\cdots & & & \cdots & & \\
0 & & & & A & 0 \\
\hline
* & 0 & \cdots & & 0 & *
\end{array} \right) \equiv \det[\frac{\partial \bar{F}^{aA} \nu_a}{\partial U^B}]. \tag{27}
$$

Here, we have used

$$RA^{-1}S \equiv \sum_{\lambda} R_{\lambda} A^{-1} S^{\lambda}. \tag{28}$$

<u>Step (b):</u> By a procedure similar to that used in Step (a), but now row-wise, we may proceed to cancel each R_λ in the leading row block through multiplication by matrices with unit determinant from the *left*. Beginning with cancellaetion of R_1,

$$D(U;\nu_a) = \det \left(\begin{array}{ccccc|c} \mathrm{id} & -R_1 A^{-1} & 0 & \cdots & 0 & 0 \\ \hline 0 & \mathrm{id} & & & & 0 \\ 0 & & \mathrm{id} & & & 0 \\ \cdots & & & \cdots & & \\ 0 & & & & \mathrm{id} & 0 \\ \hline 0 & 0 & \cdots & & 0 & \mathrm{id} \end{array} \right) [\frac{\partial \bar{F}^{aA} \nu_a}{\partial U^B}] \tag{29}$$

$$= \det \left(\begin{array}{ccccc|c} K - RA^{-1}S & 0 & R_2 & \cdots & R_N & * \\ \hline 0 & A & & & & 0 \\ 0 & & A & & & 0 \\ \cdots & & & \cdots & & \\ 0 & & & & A & 0 \\ \hline * & 0 & \cdots & & 0 & * \end{array} \right), \tag{30}$$

we may continue in this manner until all R_λ have been removed:

$$D(U;\nu_a) = \det \left(\begin{array}{ccccc|c} K - RA^{-1}S & 0 & \cdots & & 0 & * \\ \hline 0 & A & & & & 0 \\ 0 & & A & & & 0 \\ \cdots & & & \cdots & & \\ 0 & & & & A & 0 \\ \hline * & 0 & \cdots & & 0 & * \end{array} \right) \tag{31}$$

It follows that

$$D = |A|^N \det \left(\begin{array}{c|c} K - RA^{-1}S & * \\ \hline * & * \end{array} \right) \equiv |A|^N \det[D_A^B](U;\nu_a). \tag{32}$$

<u>Step (c):</u> It may be readily verified that

$$\begin{aligned} A^{-1} = & \quad -(u^c\nu_c)^{-1}[g_a^b + u^b u_a - \nu^{-2}\nu^b\nu_a], \\ \det A = & \quad \nu^2(u^c\nu_c)^2. \end{aligned} \tag{33}$$

Multiplication of $[D_A^B](U;\nu_a)$ by an appropriate matrix with unit determinant from the *right*, whereby multipling the leading column block by A from the right, leads us to consider

$$\begin{aligned} D(U;\nu_a) \quad &= |A|^{N-1} \det \left(\begin{array}{c|c} (KA - RA^{-1}SA & * \\ \hline *A & * \end{array} \right) \\ &\equiv |A|^{N-1} \det[\bar{D}_A^B](U;\nu_a). \end{aligned} \tag{34}$$

<u>Step (d):</u> It remains to ascertain that $\det[\bar{D}_A^B](U;\nu_a)$ constitutes a polynomial in ν_a. This is not immediate, in view of the presence of A^{-1} in $[\bar{D}_A^B]$ and the factor $(u^c\nu_c)^{-1}$ in (33). As it turns out, any rational factor thus introduced in $\det[\bar{D}_A^B](U;\nu_a)$ will fortuitously be cancelled. This will be shown below.

Explicit evaluation of $A^{-1}S^\lambda A$ gives

$$A^{-1}S^\lambda A = (u^c\nu_c)^{-1}[(\nu^2 h^{\lambda b} - (h^{\lambda c}\nu_c)\nu^b)u_a] + Q^\lambda, \tag{35}$$

where $Q^\lambda = (h^{\lambda c}\nu_c)\mathrm{id} + [u^b\{(u^c\nu_c)h_a^\lambda - (h^{\lambda c}\nu_c)u_a]$ represents the remaining matrix whose entries form polynomial expressions in ν_a. This gives

$$RA^{-1}SA = \nu^2(u^c\nu_c)^{-1}[(h^2\nu^b - \tilde{h}^b)u_a] + RQ, \tag{36}$$

where $RQ \equiv \sum_\lambda R_\lambda Q^\lambda$ and $\tilde{h}^b = \sum_\lambda (h^c_\lambda \nu_c) h^{\lambda b}$. This shows that $(u^c\nu_c)^{-1}$ enters in a rank-one update to the (ν_a- polynomial) $KA - RQ$, and, therefore, introduces a factor $(u^c\nu_c)^k$ with $k \geq -1$ in each term in $\det[\bar{D}^B_A](U;\nu_a)$. This follows by Lemma 4.1. More elementary, we may introduce a special frame of reference in which $u_a = (-1,0,0,0)$. $(u^c\nu_c)^{-1}$ then simply enters in the first column of $[\bar{D}^B_A](U;\nu_a)$, and the forementioned observation readily follows.

Inspection of the full matrix $[\bar{D}^B_A](U;\nu_a)$ with $U = (u^b, h^{\lambda b}, P, r)$,

$$[\bar{D}^B_A](U;\nu_a) = \left(\begin{array}{c|c|c} KA - RA^{-1}SA & q_P(u^c\nu_c)[u^b] + [\nu^b] & q_r(u^c\nu_c)[u^b] \\ \hline r[\nu_c A^c_a] & 0 & (u^c\nu_c) \\ (\xi^c\nu_c)[u_c A^c_a] & 0 & 0 \end{array} \right), \tag{37}$$

where $q_y = \partial q/\partial y$, $q = rf + h^2$, further shows that the last column is proportional to $(u^c\nu_c)$. This dependency introduces a factor $(u^c\nu_c)$ in $\det[\bar{D}^B_A](U;\nu_a)$ which, combined with the foregoing, forces each term in $\det[\bar{D}^B_A](U;\nu_a)$ to contain $(u^c\nu_c)^k$ with $k \geq 0$: $\det[\bar{D}^B_A](U;\nu_a)$ forms a polynomial expression in ν_a (homogeneous of degree 10).

Together, the results from Step (c) and Step (d) show that $|A|^{N-1}$ is a factor in the the $6+4N$-degree homogeneous polynomial $D(U;\nu_a)$. $\square$

The full factorization is now contained in

Theorem 4.1 *Yang-Mills' characteristic determinant is of the form*

$$D(U;\nu_a) = -(\xi^c\nu_c)(u^c\nu_c)^{2N-1}(\nu^c\nu_c)^N Y(U;\nu_a), \tag{38}$$

where $Y(U;\nu_a)$ is a sixth-order polynomial, homogeneous in ν_a.

Proof : Inspection of $[\bar{D}_A^B](U;\nu_a)$ as defined in the proof of Lemma 4.1 shows that all its entries are even in u^b except for the lower row

$$[r[\nu_c A^c], 0, (u^c\nu_c)] \tag{39}$$

stemming from conservation of baryon number, $\nabla_a(ru^a)=0$. det $[\bar{D}_A^B](U;\nu_a)$ is therefore odd in u^b. In view of $|A|=(\nu^c\nu_c)(u^c\nu_c)^2$, this forces $D(U;\nu_a)$ to be odd in u^b.[1] Now notice that u^b appears only through $(u^c\nu_c)$ in the scalar field $D(U;\nu_a)$, since $u^c h_c^\lambda=0$ for all λ and $u^c u_c=-1$ (see also Proposition 5.1 in [**?**]). The roots of $D(U;\nu_a)$ are invariant quantities, necessarily invariant under a sign-change $u^b\rightarrow -u^b$. It follows that an odd factor of $(u^c\nu_c)$ must factor in $D(U;\nu_a)$, leaving all other factors even in $(u^c\nu_c)$. The factor $(u^c\nu_c)^{2N-2}$ introduced by $|A|^{N-1}$, therefore, forces $(u^c\nu_c)^{2N-1}$ to be a factor in det $D(U;\nu_a)$.

The factor $(\nu^c\nu_c)^N=\nu^{2N}$ in (38), rather than ν^{2N-2} as promised by $|A|^{N-1}$, follows from the fact that the $6\times 4N$ dimensional elements

$$[0_a, \mu_1\nu_a, \cdots, \mu_N\nu_a, 0, 0], \tag{40}$$

$\mu_k\epsilon\mathcal{C}$, form left null elements of the characteristic matrix $[F_A^B](U:\nu_a)$, whenever $\nu^2=0$. This follows from ν_a being a null element of *both* A_a^b and $S_a^{\lambda b}$, *i.e.*,

$$\nu_c A_a^c = \nu_c S_a^{\lambda c} = 0, \tag{41}$$

whenever $\nu^2=0$. This completes the proof.□

[1]In [**?**] the characteristic determinant of ideal MHD ($N=1$) is even in u^b, as we work there with $u^a\nabla_a S=0$ (instead of $\nabla_a\{(u^2+1)\xi^a\}=0$).

We conclude that the null cone $\mathcal{N}$ at each point on the space-time manifold is described by

$$\begin{array}{ll} (1)\text{ Entropy waves:} & u^c\nu_c = 0, \\ (2)\text{ Yang-Mills waves:} & Y(U;\nu_a) = 0. \end{array} \tag{42}$$

Thus, the Yang-Mills waves are completely defined by a sixth-order polynomial $Y(U;\nu_a)$.

Recall that in the theory of ideal magneto-hydrodynamics (MHD) we find the hydrodynamical waves and the Alfven waves through the zeros of a sixth order polynomial, which factorizes over a fourth-order (hydrodynamical waves) and a second-order (Alfven waves) polynomial [**?**, **?**]. However, one may not expect such factorization to persist in the more general case of a Yang-Mills fluid.

Expressions (27)-(28) show that the Yang-Mills magnetic stress-tensor H_a^b, as defined above, represents the entire magnetic field $h_a \epsilon \mathcal{G}$ in the infinitesimal wave structure. It follows that the Yang-Mills waves $Y(U;\nu_a) = 0$ are completely determined by (u^a, H_a^b, P, r). Because H_{ab} is real-symmetric, the Spectral Decomposition Theorem allows for the representation

$$H_{ab} = \sum_{i=1}^{3} s_a^{(i)} s_b^{(i)}, \tag{43}$$

the $s_a^{(i)}$, $i = 1, 2, 3$, being three one-forms on the space-time manifold. Since $H_{ab}u^a \equiv 0$, it follows that $N = 3$ without loss of generality in the structure of the infinitesimally small amplitude waves in ideal Yang-mills fluids.

The new wave features in Yang-Mills fluids in ideal Yang-Mills fluids are contained in the normal cone as defined by the zeros of the *Yang-Mills characteristic polynomial* $Y(U;\nu_a)$. Of foremost importance is to establish hyperbolicity, *i.e.*, to establish that $Y(U;\nu_a)=0$ contains a full set of three (real-valued) normal conical sheats which are everywhere space-like. Of course, we need not require strict, algebraic hyperbolicity, that is, some of these normal sheats may (partially) coincide, associated with multiplicities of the roots of $Y(U;\nu_a)=0$. This may be studied through symmetrization.

5 On hyperbolicity of systems of conservation laws

Friedrichs & Lax [**?**] and Friedrichs [**?**] conceived the deep and general result that a system of conservation laws is well-posed when the system (implicitly) contains a conservation law for a convex quantity. A quantity of this kind naturally enforces an integral bound on the solution for all time. More specific regularity results for finite time are obtained following symmetrization, as obtained by such conserved convex quantity. The resulting system is in *symmetric hyperbolic* form ([**?**]), which establishes at one stroke both hyperbolicity and well-posedness. In our study of the Yang-Mills equations, a slightly weaker version of Friedrichs' symmetrization procedure is formulated in view of the conserved, algebraic constraints in the theory of ideal Yang-Mills fluids.

5.1 Friedrichs' symmetrization procedure

Ideal Yang-Mills as given in Theorem 3.1 is a particular case of a system of M conservation laws of the form

$$\nabla_a F^{aB} = f^B. \tag{44}$$

Friedrichs [**?**] introduced certain Properties CI and CII, discussed below, which are sufficient for (44) to constitute a *symmetric hyperbolic* system,

$$A^{aAB}(V)\nabla_a V_A = A^{tAB}(V)\nabla_t V + A^{\alpha AB}(V)\nabla_\alpha V_A = f^B(V), \tag{45}$$

characterized by symmetry of the A^{aAB} (in A and B) and positive definiteness of A^{tAB}. Systems of this type are known to give rise to well-posed initial value with sensitivity on initial data of finite order. In the context of general relativity, the requirement that $A^{aAB}\xi_a$ is nonsingular for all time-like one-forms ξ_a furthermore ensures that no infinitesimally small amplitude wave exceeds unit velocity.

Friedrichs' [**?**] symmetrization procedure applies to systems possessing what Friedrichs calls a "main dependency" relation of the form

$$\mathrm{CI}: \quad W_A \delta F^{aA} \equiv 0. \tag{46}$$

Here, W_A is a nontrivial vector and δ denotes a total variation (this property is called CI$'$ in [Friedrichs, 1974]). Evidently, this requires $W_A f^A \equiv 0$. Then differentiation with respect to any V^A yields

$$\frac{\partial W_A}{\partial V^C}\frac{\partial F^{aA}}{\partial V^D}\nabla_a V^D + W_A \frac{\partial^2 F^{aA}}{\partial V^C \partial V^D}\nabla_a V^D = 0. \tag{47}$$

This shows that for each a

$$\frac{\partial W_A}{\partial V^C}\frac{\partial F^{aA}}{\partial V^D} \tag{48}$$

is symmetric (in C and D). If, furthermore, V^A is such that for some time-like ξ_a

$$\text{CII:}\quad \delta W_A \delta F^{aA}\xi_a = \delta V^C\left(\tfrac{\partial W_A}{\partial V^C}\tfrac{\partial F^{aA}\xi_a}{\partial V^D}\right)\delta V^D \tag{49}$$

is positive definite (for all δV^A), then (44) naturally constitutes a symmetric hyperbolic system in V_A in the form of

$$\frac{\partial W_A}{\partial V^C}\frac{\partial F^{aA}}{\partial V^D}\nabla_a V^D = \frac{\partial W_A}{\partial V^C} f^A. \tag{50}$$

Writing $Z^A = \nabla_a F^{aA} - f^A$, it follows that solutions to (50) satisfy

$$\left\{\begin{array}{l} \frac{\partial W_A}{\partial V^C} Z^A = 0, \\ W_A Z^A = 0, \end{array}\right. \tag{51}$$

where the second equation is a consequence of Property CI. If Property CII is satisfied as well, $(W_A, \frac{\partial W_A}{\partial V^C})$ can be readily seen to be nonsingular, so that any solution to (51), is, in fact, a solution to (44).

This summarizes the symmetrization procedure constructed by Friedrichs [?] for systems possessing a single dependency relation (46). It should be mentioned that Friedrichs [?] also treats the case of more than one dependency relations. For the purpose of symmetrization of ideal Yang-Mills in divergence form there exists precisely one, main dependency relation, although in a slightly weaker form than CI as stated above. Not suprisingly, this is due to the fact that the divergence formulation allows for a larger class of solutions, which includes the nonphysical solutions with $c_\lambda \neq 0$.

5.2 Symmetrization in the presence of conserved constraints

Ideal Yang-Mills in divergence form contains the constraints $c_\lambda = 0$ as conserved quantities when initial data satisfy certain compatibility conditions. The system itself allows for solutions with $c_\lambda \neq 0$ as well, in response to nonphysical initial data. It is therefore not suprising that in this generality ideal Yang-Mills in divergence form does not satisfy conditions CI or CII. Rather, we require a main dependency relation of the form

$$\mathrm{YI}: \quad W_A \delta F^{aA} \equiv \delta z^a \tag{52}$$

for some nontrivial W_A and some vector field z^a. We will later require that z^a vanishes for solutions which satisfy the constraints. Clearly, Property YI is weaker than Friedrichs' Property CI. However, the identity

$$\frac{\partial W_A}{\partial V^C}\frac{\partial F^{aA}}{\partial V^D}\nabla_a V^D + W_A \frac{\partial^2 F^{aA}}{\partial V^C \partial V^D}\nabla_a V^D = \frac{\partial^2 z^a}{\partial V^C \partial V^D}\nabla_a V^D, \tag{53}$$

instead of (47), clearly does not change the symmetry of the coefficient matrices (48). In what follows, nonsingularity of $(W_A, \frac{\partial W_A}{\partial V^C})$ will be assumed, so that (51) enforces equivalence of (53) with (44).

For solutions with $c_\lambda \neq 0$, different from the Yang-Mills solutions $c_\lambda = 0$, we may not expect convexity property CII to hold. We do, however, wish solutions to ideal Yang-Mills satisfying the constraints

$$c_0 = 0, \tag{54}$$

$$c_\lambda = 0, \tag{55}$$

to satisfy a symmetric hyperbolic system. To this end, a slightly weaker formulation of CII will be used.

Definition 5.1 *Consider a scalar* $Q = Q(U^B, \delta U^C)$ *and a set of scalar constraints,* $c_k(U^C) = 0$*. The variation* $\delta U^C \neq 0$ *satisfying* $\delta c_k(U) = 0$ *will be referred to as the* **constraint variation** *of* U^C *with respect to the* $c_k = 0$*. If* $Q > 0$ *with respect to the constraint variation of* U^C*, then* Q *is said to be* **constraint positive definite** *with respect to the* $c_k = 0$*.*

Notice that for each U^B the constraints in Definition 5.1 define a linear subspace of variations δU^C.

We now define constraint positive definiteness of

$$\text{YII:} \quad Q := \delta W_A \delta F^{aA} \xi_a = \delta V^C \left(\frac{\partial W_A}{\partial V^C} \frac{\partial F^{aA} \xi_a}{\partial V^D}\right) \delta V^D \tag{56}$$

for some time-like ξ_a as Property YII. Clearly, YII is weaker than Friedrichs' Property CII.

Property YI and YII are sufficient for (44) to imply a symmetric hyperbolic system in V_A. To see this, we use the following construction.

Lemma 5.1 *Given a real-symmetic* $A \epsilon \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$ *which is positive definite on a linear subspace* $\mathcal{V} \subset \mathbf{R}^n$*, there exists a real-symmetric, positive definite* $A^* \epsilon \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$ *such that*

$$A^* y = Ay \ (y \epsilon \mathcal{V}). \tag{57}$$

Proof : Let $\mathcal{V}^\perp$ denote the orthogonal complement of $\mathcal{V}$. If $\mathcal{V}^\perp = \{0\}$, we may take $A^* = A$. Otherwise, pick $x \epsilon \mathcal{V}^\perp$ with $||x|| = 1$, and form

$$A_1 = A + \mu x x^T, \tag{58}$$

where x^T denotes the transpose of x. A_1 is manisfestly symmetric. We will choose μ so that A_1 is positive definite on the space $\mathcal{V}_1$ of all vectors of the form $z = y + \lambda x$, where $y \epsilon \mathcal{V}$ and λ is a real scalar.

To this end, evaluate

$$\begin{aligned} z^T A z &= y^T A y + 2\lambda x^T A y + \lambda^2 x^T A x + \lambda^2 \mu^2 \\ &\geq c||y||^2 - 2|\lambda| M ||y|| + \lambda^2(\mu^2 - M) \\ &= (\mu^2 - M)(|\lambda| - \tfrac{M||y||}{\mu^2 - M})^2 + c||y||^2 - \tfrac{M^2}{\mu^2 - M}||y||^2. \end{aligned} \tag{59}$$

Here, $c > 0$ is a coercivity constant for A on $\mathcal{V}$, *i.e.*, $y^T A y \geq c||y||$ $(y \epsilon \mathcal{V})$, and $M = ||A||$. For sufficiently large $\mu > M^{1/2}$, we have

$$c||y||^2 - \frac{M^2}{\mu^2 - M}||y||^2 \geq \frac{c}{2}||y||^2. \tag{60}$$

Writing $\lambda = a||y||$, (59) yields

$$z^T A z \geq K(a)||z||^2, \tag{61}$$

where $K(a) = (1 + a^2)^{-1}\{(\mu^2 - M)(a - \frac{M}{\mu^2 - M})^2 + \frac{c}{2}\}$. Clearly, $\inf_{a \epsilon \mathbf{R}} K(a) \equiv c_1 > 0$.

We have now proved that there exists an embedding of $A|_{\mathcal{V}}$ in a symmetric A_1 which is positive definite on $\mathcal{V}_1$.

Repeating the construction above $k = \dim \mathcal{V}^{\perp}$ times, we may exhaust $\mathcal{V}^{\perp}$, thereby arriving at a symmetric $A^* = A_k$ which is positive definite on $\mathcal{V}_k = \mathbf{R}^n$. This completes the proof. $\square$

Consider a system (44) which satisfies Property YI, thereby arriving at a system (45), which may, alternatively, be written as

$$A^{tAB}(V)\partial_t V + A^{\alpha AB}(V)\partial_\alpha V_A = \bar{f}^B(V). \tag{62}$$

Here, $\bar{f}^B$ now contains the contributions from the connection symbols as well. Assuming that (44) satisfies Property YII with $\xi^b = (\partial_t)^b$, we may form a positive definite $(A^{tAB})^*$ following Lemma 5.1. Indeed, for V such that $c_\lambda(V) = 0$ for all λ, Property YII implies $A^{tAB}(V)$ is positive definite on $\mathcal{V} = \bigcap_\lambda \{\delta V | \frac{\partial c_\lambda}{\partial V^C} \delta V_C = 0\}$. According to Lemma 5.1, we may insist

$$(A^{tAB})^* y = A^{tAB} y \tag{63}$$

for all constraint variations $y \epsilon \mathcal{V}$.

We are thus led to consider the symmetric hyperbolic system

$$(A^{tAB})^*(V)\partial_t V + A^{\alpha AB}(V)\partial_\alpha V_A = \bar{f}^B(V). \tag{64}$$

The arguments above show that solutions to the original divergence system of equations (44) in response to initial data compatible with (10) (which have the property that $c \equiv 0$ throughout) are solutions to the symmetric hyperbolic system of equations (64). It naturally follows that these solutions depend continuously on their data as described in [**?**].

Friedrichs's symmetrizations procedure and the symmetrization procedure in the presence of conserved constraints as presented in this Section are summarized in Table I.

It would be of interest to also use the symmetric hyperbolic system of equations (64) for existence proofs of solutions to the divergence formulation of ideal Yang-Mills fluids. To this end, solutions to initial value problems for the symmetric hyperbolic system of equations (64) in response to data compatible with (10) must be shown to have the property that $c \equiv 0$ throughout. However, this falls outside the scope of this work.

Friedrichs' procedure	*In this paper, with conserved $c = 0$*
Properties Cx	*Properties Yx*
CI: $W_A \delta F^{aA} = 0$ CII: $\delta W_A \delta F^{aA} \xi_a$ positive definite (CIII when more than one dependency relation [?])	YI: $W_A \delta F^{aA} = \delta z^a$ YII: $\delta W_A \delta F^{aA} \xi_a$ constraint positive definite YIII: $c \equiv 0$ if (10) satisfied
$\frac{\partial W_A}{\partial V^C} \frac{\partial F^{aA}}{\partial V^D} \nabla_a V^D = \frac{\partial W_A}{\partial V^C} f^A$	$(\frac{\partial W_A}{\partial V^C} \frac{\partial F^{tA}}{\partial V^D})^* \partial_t V^D + \frac{\partial W_A}{\partial V^C} \frac{\partial F^{\alpha A}}{\partial V^D} \partial_\alpha V^D = \frac{\partial W_A}{\partial V^C} \bar{f}^B$

Table 1: Symmetrization procedures: summary of Friedrichs' symmetrization procedure and the symmetrization procedure in the presence of conserved constraints, as presented in this paper. Friedrichs' procedure requires a system $\nabla_a F^{aB} = 0$ to possess Properties CI and CII (in the presence of a single dependency relation), and the procedure presented here requires the system to possess Properties YI, YII and YIII.

6 Ideal Yang-Mills in symmetric hyperbolic form

We will show that ideal Yang-Mills satisfies a symmetric hyperbolic system of the form (45). Using the symmetrization procedure in the presence of constraints as outlined in the previous section, this result follows by showing that ideal Yang-Mills in divergence form (11) satisfies Properties YI and YII. The proof of Theorem 3.1 establishes that (11) satisfies Property YIII. It is well-known that the equations of relativistic hydrodynamics can be written in symmetric hyperbolic form [?, ?]. The desired result, therefore, will be established "bootstrap-wise," by showing that ideal Yang-Mills can be written in symmetric hyperbolic form, whenever the equations of relativistic hydrodynamics can be written in symmetric hyperbolic form.

For the purpose of symmetrization, we use a second divergence form of ideal Yang-Mills fluids (following [?]), in which the equations of hydrodynamics are more similar to those already discussed in the literature:

$$\hat{\nabla}_a F^{aA} \equiv \begin{cases} \nabla_a T^{ab} = 0, \\ \hat{\nabla}_a \{h^{[a}u^{b]} + g^{ab}c(U)\} = 0, \\ \nabla_a (ru^a) = 0, \\ \nabla_a (rSu^a) = 0. \end{cases} \tag{65}$$

This system may readily be seen to be equivalent to (11) for continously differentiable solutions in view of the thermodynamic relation $\mathrm{d}P = r\mathrm{d}f - rT\mathrm{d}S$. Note, however, that the jump conditions across surfaces of discontinuity as follow from a weak formulation for this system (65) are improper (see [?] for a discussion on this point in magneto-hydrodynamics). We organise (65) as follows

$$\nabla_a F_f^{aB} + \nabla_a F_m^{aB} = f_f^B + f_m^B \tag{66}$$

where

$$\begin{aligned} &\nabla_a F_f^{aB} \equiv \begin{cases} \nabla_a T_f^{ab}, \\ \nabla_a (ru^a), \\ \nabla_a \{\xi^a (u^2+1)\}, \end{cases} && f_f^B = 0, \\ &\nabla_a (F_m^{aA})_\lambda \equiv \begin{cases} \nabla_a T_m^{ab}, \\ \nabla_a \omega_\lambda^{ab}, \end{cases} && f_m^B = \begin{cases} 0, \\ -c_{\lambda\mu\nu} A_e^\mu \omega^{\nu eb}. \end{cases} \end{aligned} \tag{67}$$

Here, $c_{\lambda\mu\nu}$ are the structure constants of the Yang-Mills Lie algebra, $\mathcal{G}$. As mentioned before, these equations must be closed by (12),

$$\nabla_a A_b - \nabla_b A_a = F_{ab} - [A_a, A_b] \equiv G_{ab}. \tag{68}$$

Before turning to the symmetrization of the nonlinear system of equations (65), we wish to remark that with temporal gauge the closure equations can be seen explicitly to pose no difficulties in the symmetrization process. To see this, let ξ^b is any given time-like vector field, and consider the equations

$$(\xi^c \nabla_c) A_a - \xi^c \nabla_a A_c = \xi^c G_{ca}. \tag{69}$$

Because $\hat{\nabla}_a \omega^{ab} = 0$ enforces $\hat{\nabla}_{[a} F_{bc]} = \nabla_{[a} F_{bc]} + [A_{[a}, F_{bc]}] = 0$, and $[F_{[ab}, A_{c]}] \equiv \nabla_{[a}[A_b, A_{c]}]$, the right hand-side in (68) satisfies $\nabla_{[a} G_{bc]} = 0$ (using the result that $c(U) = 0$). Denoting with Greek subscripts contractions with ξ^b-orthogonal vectors, (69) thus may be seen to ensure conservation of $\nabla_\alpha A_\beta - \nabla_\beta A_\alpha = G_{\alpha\beta}$. Eqs. (11)-(12) therefore impose (75) as evolution equations for A_a. With conservation $\xi^c A_c = 0$, any solution to (69) satisfies

$$(\xi^c \partial_c) A_a = \xi^c G_{ca} + A_c \partial_a \xi^c \tag{70}$$

with coefficient matrices

$$(\xi^c \nu_c) g_a^b. \tag{71}$$

For this reason, the closure equations (12) with temporal gauge need no further consideration in the symmetrization of the equations of ideal Yang-Mills in divergence form.

In the variables (u^b, h^b, f, S) (65) is, of course, regular in the sense of Cauchy-Kowalewski (by its equivalence to (11)), and possesses therefore no dependency relation. If we consider (65) in $V = (v_\alpha, h^b, f, S)$, however, where v_α is

a three-parametrization of u^b such that $u^2 \equiv -1$, this system becomes overdetermined by one and a main dependency relation results (as in Property YI). This will be made explicit below.

6.1 Symmetrization of hydrodynamics

The equations of relativistic hydrodynamics,

$$\nabla_a F_f^{aB} = 0, \tag{72}$$

as follow from ideal Yang-Mills in divergence form with vanishing Yang-Mills field, may be written in symmetric hyperbolic form following the arguments given by Friedrichs [?], Ruggeri & Strumia [?] and Annile [?]. Friedrichs [?] emphasis that symmetric hyperbolic systems will not generally be in covariant form; the construction of systems of this type is intended as a method of establishing well-posedness, an aspect of which *is* frame independent. For symmetrization of the equations of relativistic hydrodynamics, the velocity four-vector, u^b, may be represented by a three-parameter representation, $u^b = u^b(v_\alpha)$, in view of $c_0 = u^c u_c + 1 = 0$. Consider $W_A^f = \frac{1}{T}(u_a, f - TS, T)$ as a function of $V_C^f = (v_\alpha, T, f)$. Notice that variations δV_A^f satisfy $\delta u^2 = 0$ by construction. In this parametrization, both Friedrichs' Properties CI & CII hold true. Indeed, we have a main dependency relation and an associated quadratic, Q_f, of the form [?]

$$\begin{array}{lcl} W_A^f \delta F_f^{aA} & = & u_b \delta T_f^{ab} + (f - TS)\delta(ru^a) + T\delta(rSu^a) \equiv 0, \\ Q_f := \delta W_A \delta F_f^{aA} & = & T\{\delta[\frac{u_b}{T}]\delta T_f^{ab} + \delta[\frac{f-TS}{T}]\delta(ru^a)\} + \frac{\delta T}{T} W_A^f \delta F_f^{aA} \\ & \equiv & T\{\delta[\frac{u^b}{T}]\delta T_f^{ab} + \delta[\frac{f-TS}{T}]\delta(ru^a)\}. \end{array} \tag{73}$$

Ruggeri & Strumia [?] and Anile [?] have shown Q_f to be positive definite, provided that the following conditions on the free enthalpy, G, and the sound

velocity, a, are satisfied [**?**]:

$$\left\{ \begin{array}{ll} \text{the free enthalpy:} & -G(T,P) \text{ is convex,} \\ \text{the sound velocity:} & a^{-2} = \frac{f}{r}\frac{\partial r}{\partial f}|_S > 1. \end{array} \right. \tag{74}$$

We will now proceed with the full equations of ideal Yang-Mills in divergence form.

6.2 Symmetrization with Yang-Mills field

Ideal Yang-Mills in divergence form will be shown to be satisfying Property YI and YII, provided that the hydrodynamical conditions (74) have been met.

Property YI. Recall from (5) and (14) the contribution to the stress-energy tensor by the magnetic field, T^{ab}_m, and the new Yang-Mills tensor, ω^{ab}, respectively, given by

$$T^{ab}_m = h^2 u^a u^b + \frac{h^2}{2} g^{ab} - h^a h^b, \tag{75}$$

$$\omega^{ab} = h^a u^b - u^a h^b + g^{ab} h^c u_c. \tag{76}$$

Let us consider $W_A = \frac{1}{T}(u_a, h_a, f)$ with parametrization $V_A = (v_\alpha, h_a, T, f)$. The total variation, δ, then satisfies $c_0(U) := u^2 + 1 \equiv 0$ by construction, as before. The variational expressions

$$\begin{array}{rcl} u_b \delta T^{ab}_m & = & u_b\{h^2 u^a \delta u^b + h^2 u^b \delta u^a + 2u^a u^b h_c \delta h^c \\ & & +g^{ab} h_c \delta h^c - h^b \delta h^a - h^a \delta h^b\} \\ & = & -h^2 \delta u^a - u^a (h_c \delta h^c) - h^a (u_c \delta h^c) - c\delta h^a, \\ h_b \delta \omega^{ab} & = & h_b\{h^a \delta u^b + u^b \delta h^a - h^b \delta u^a - u^a \delta h^b + g^{ab} \delta c\} \\ & = & h^a (h_c \delta u^c) + c\delta h^a - h^2 \delta u^a - u^a (h_c \delta h^c) + h^a \delta c, \end{array} \tag{77}$$

introduce the identity

$$u_b \delta T^{ab} - h_b \delta \omega^{ab} = -2\delta(c h^a) \equiv \delta z^a. \tag{78}$$

This establishes that Property YI is satisfied:

$$W_A \delta(F_f^{aA} + F_m^{aA}) = \delta z^a. \tag{79}$$

Property YII. Ideal Yang-Mills in divergence form can be shown to be satisfying Property YII by consideration of the extension to relativistic hydrodynamics as introduced by the Yang-Mills fields.

Proposition 6.1 *If the equations of relativistic hydrodynamics are convex, then ideal Yang-Mills in divergence form is constraint convex.*

Proof : Let Q_f denote the quadratic variation associated with the equations of relativistic hydrodynamics as in (73), and suppose conditions (74) have been met.

Consider the variation, δ, introduced by the total variation of V_C (*i.e.*, variations satisfying $\delta c_0(U) = \delta(u^2+1) \equiv 0$). The variational expressions

$$\begin{array}{rcl} \delta u_b \delta T_m^{ab} \xi_a & = & h^2(u^c\xi_c)(\delta u_c \delta u^c)^2 \\ & & +(\xi_c \delta u^c)(h_c \delta h^c) - (h_c \delta u^c)(\xi_c \delta h^c) - (\xi_c h^c)(\delta u_c \delta h^c), \\ \delta h_b \delta \omega^{ab} \xi_a & = & (h^c \xi_c)(\delta h_c \delta u^c) + (\xi_c \delta h^c)(u_c \delta h^c) \\ & & -(\xi_c \delta u^c)(h_c \delta h^c) - (u^c \xi_c)(\delta h_c \delta h^c) + (\xi_c \delta h^c)\delta c, \end{array} \tag{80}$$

introduce the identities

$$\begin{array}{rcl} \delta u_b \delta T_m^{ab} \xi_a - \delta h_b \delta \omega^{ab} \xi_a & = & (u^c\xi_c)[h^2(\delta u)^2 + (\delta h)^2] + 2[(\xi_c \delta u^c)(h_c \delta h^c) \\ & & -(h^c \xi_c)(\delta u_c \delta h^c)] - 2(\xi_c \delta h^c)\delta c \equiv Q_0. \end{array} \tag{81}$$

If δh^b varies freely, disregarding $\delta c_\lambda = 0$, the quadratic Q_0 fails to be positive definite.

In the presence of the constraint variations with respect to the $c_\lambda = 0$, the last term in the expression for Q_0 in (81) vanishes identically:

$$-2(\xi_c \delta h^c)\delta c = \sum_\lambda -2(\xi_c \delta h_\lambda^c)\delta c_\lambda \equiv 0, \tag{82}$$

and we are left with

$$\begin{aligned} Q_m: \ &= \delta u_b \delta T_m^{ab} \xi_a - \delta h_b \delta \omega^{ab} \xi_a \\ &= (u^c \xi_c)[h^2(\delta u)^2 + (\delta h)^2] + 2[(\xi_c \delta u^c)(h_c \delta h^c) - (h^c \xi_c)(\delta u_c \delta h^c)]. \end{aligned} \tag{83}$$

This expression *is* (constraint) positive definite in constraint variations with $\delta h \neq 0$. To see this, it is convenient to work in a local Lorenzian frame of reference $g^{ab} = \eta^{ab} = \text{dia}(-1,1,1,1)$ with $\{x^a\} = (x^0, x^\alpha)$, $\alpha = 1,2,3$, such that $u^b = (1,0,0,0)$. The magnetic field, h_λ^b, and the variations δu^b and δh_λ^b may then be expressed in terms of three-component, spatial vectors, k_λ^β, δv^β and δk^β, respectively, as

$$\begin{aligned} &h_\lambda^b = k_\lambda^\beta \eta_\beta^b, \\ &\delta u^b = \delta v^\beta \eta_\beta^b, \\ &\delta h_\lambda^b = -u^b \delta\epsilon_\lambda + \delta k_\lambda^\beta \eta_\beta^b. \end{aligned} \tag{84}$$

Notice that $\delta\epsilon_\lambda$ follows from $\delta c_\lambda = \delta\epsilon_\lambda + k_\lambda^\mu \delta v_\mu = 0$. In the constraint variation of δ, the right hand-side in the quadratic variation in (81) becomes a $\lambda-$sum of quadratic forms $\delta W_{A\lambda} Q_\lambda^{AB} \delta W_{B\lambda}$ in $\delta W_{A\lambda} = (\delta v_\alpha, \delta k_{\alpha\lambda})$, where

$$Q_\lambda^{AB} = \begin{pmatrix} (u^c\xi_c)[k^2\eta^{\alpha\beta} - k_\lambda^\alpha k_\lambda^\beta] & [\xi^\alpha k_\lambda^\beta - (k_\lambda^\mu \xi_\mu)\eta^{\alpha\beta}] \\ [k_\lambda^\alpha \xi^\beta - (k_\lambda^\mu \xi_\mu)\eta^{\alpha\beta}] & (u^c\xi_c)[\eta^{\alpha\beta}] \end{pmatrix}. \tag{85}$$

With $(u^c\xi_c) = \xi_0 > 0$, the six eigenvalues, s_l, of Q_λ^{AB} may be evaluated as

$$\begin{cases} s_1 = 0, \\ s_2 = (u^c\xi_c) > 0, \\ s_3^\pm = A \pm (A^2 - [(u^c\xi_c)^2 k^2 - (k^c\xi_c)^2])^{\frac{1}{2}} > 0, \\ s_4^\pm = A \pm (A^2 - k^2|\xi^2|)^{\frac{1}{2}} > 0, \end{cases} \tag{86}$$

where $A = \frac{1}{2}(u^c\xi_c)(k^2+1)$. Here, we have used $\xi^2 = -\xi_0^2 + \xi^\alpha\xi_\alpha < 0$, so that $\xi_0^2 = |\xi^2| + \xi^\alpha\xi_\alpha \geq |\xi^2|$ and $\xi_0^2 > \xi_\alpha^2$ for each $\alpha = 1,2,3$. Notice that the eigenvector of Q_λ^{AB} associated with $s_1 = 0$ is $(k^\beta, 0)$. Therefore, $W_{A\lambda} Q_\lambda^{AB} W_{B\lambda}$

is positive semi-definite, and positive definite whenever $\delta k^b \neq 0$. The sum

$$\sum_\lambda W_{A\lambda} Q_\lambda^{AB} W_{B\lambda}, \tag{87}$$

therefore, naturally shares the same properties.

If Q_f is positive definite with respect to δV_A^f, the preceeding result shows that

$$\delta V_A \delta U^A = Q_f + Q_m \tag{88}$$

is constraint positive definite with respect to the $c_\lambda = 0$. $\Box$

The constraint symmetrization procedure from Section 5.2 may now be applied to ideal Yang-Mills in divergence form with V_A as given above. It remains to ascertain the orthogonality requirement

$$\begin{aligned} W_A f^A &= h_c f_m^c = -h_c^\lambda c_{\lambda\mu\nu} A_e^\mu \omega^{\nu ec} \\ &= (u^c h_c^\lambda) c_{\lambda\mu\nu} h^{\nu e} A_e^\mu - (h_c^\lambda h^{\nu c}) c_{\lambda\mu\nu} (A_e^\mu u^e) \\ &= c^\lambda (h^{\nu e}) A_e^\mu c_{\lambda\mu\nu} = 0, \end{aligned} \tag{89}$$

where we have used total anitsymmetry of the structure constants. This completes our symmetrization procedure for ideal Yang-Mills fluids.

In closing, we remark that the simple structure of the flux-freezing constraints allows for explicit three-parametrizations

$$h_\lambda^b = -\frac{(k_\lambda^c u_c)}{(\xi^c u_c)} \xi^b + k_\lambda^b \tag{90}$$

for each Lie-component of the magnetic field, where the three-parametrized Lie-component k_λ satisfies $k_\lambda^c \xi_c \equiv 0$, which are such that $c_\lambda \equiv 0$. By suitable choice of coordinates, $\xi^b = (1, 0, 0, 0)$ so that $k_a = (0, k_\alpha)$. Our foregoing analysis shows that in the $5 + 3N$ system parametrization $V' = (v_\alpha, k_\alpha, f, T)$

is such that Friedrichs' Properties CI and CII are satisfied (with the same $W_A = (v_\alpha, h_a, f - TS, T)$ as before, but now in this reduced parametrization and with respect to all variations in this parametrization). Thus, in the case of such simple constraints, we could resort to pursuing Friedrichs' symmetrization procedure. We are then left with the task of proving that the symmetric hyperbolic system as defined by Friedrichs symmetrization procedure actually solves the entire system, now overdetermined by $N + 1$. Further dependency relations must be invoked, namely those resulting from antisymmetry of $h_{[a}u_{b]}$. We have not set out to pursue this, to see if such specific treatment results in shorter arguments, although this should lead to a successful argument as well.

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